Integration by Substitution - Past Exam Questions

1. **(Question 3 - C4 June 2017)**

3.

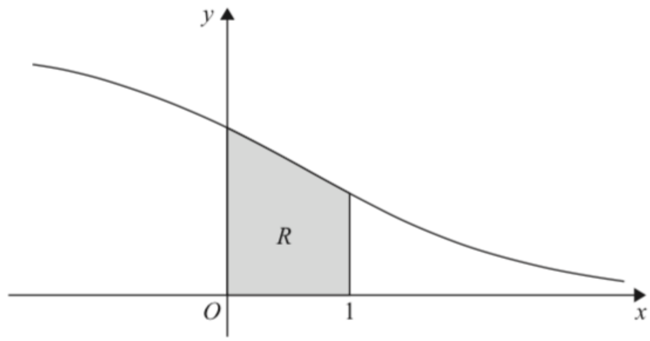

Figure 1

Figure 1 shows a sketch of part of the curve with equation $y = \dfrac{6}{(e^x + 2)}$, $x \in \mathbb{R}$

The finite region R, shown shaded in Figure 1, is bounded by the curve, the y-axis, the x-axis and the line with equation $x = 1$

The table below shows corresponding values of x and y for $y = \dfrac{6}{(e^x + 2)}$

x	0	0.2	0.4	0.6	0.8	1
y	2		1.71830	1.56981	1.41994	1.27165

(a) Complete the table above by giving the missing value of y to 5 decimal places.
(1)

(b) Use the trapezium rule, with all the values of y in the completed table, to find an estimate for the area of R, giving your answer to 4 decimal places.
(3)

(c) Use the substitution $u = e^x$ to show that the area of R can be given by

$$\int_a^b \frac{6}{u(u+2)}\, du$$

where a and b are constants to be determined.
(2)

(d) Hence use calculus to find the exact area of R.
[*Solutions based entirely on graphical or numerical methods are not acceptable.*]
(6)

2. **(Question 7 - C4 June 2016)**

7. (a) Find

$$\int (2x-1)^{\frac{3}{2}}\,dx$$

giving your answer in its simplest form.

(2)

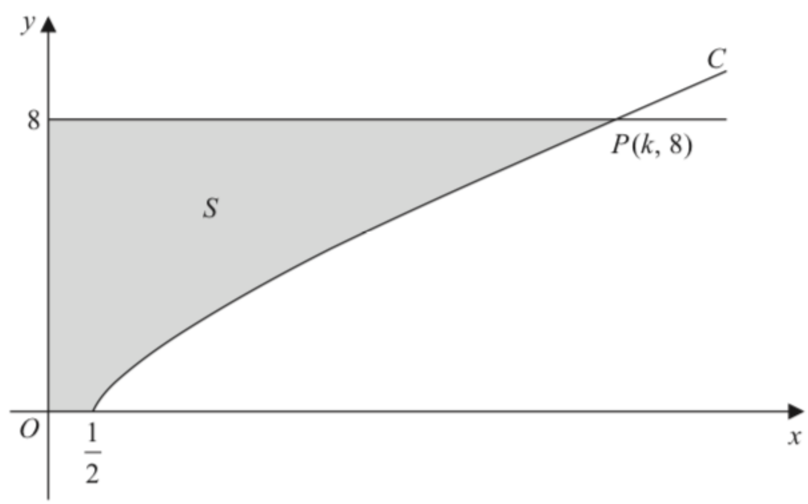

Figure 3

Figure 3 shows a sketch of part of the curve C with equation

$$y = (2x-1)^{\frac{3}{4}}, \qquad x \geqslant \frac{1}{2}$$

The curve C cuts the line $y = 8$ at the point P with coordinates $(k, 8)$, where k is a constant.

(b) Find the value of k.

(2)

The finite region S, shown shaded in Figure 3, is bounded by the curve C, the x-axis, the y-axis and the line $y = 8$. This region is rotated through 2π radians about the x-axis to form a solid of revolution.

(c) Find the exact value of the volume of the solid generated.

(4)

3. **(Question 6 - C4 June 2016)**

6. (i) Given that $y > 0$, find

$$\int \frac{3y-4}{y(3y+2)}\,dy$$

(6)

(ii) (a) Use the substitution $x = 4\sin^2\theta$ to show that

$$\int_0^3 \sqrt{\left(\frac{x}{4-x}\right)}\,dx = \lambda \int_0^{\frac{\pi}{3}} \sin^2\theta\,d\theta$$

where λ is a constant to be determined.

(5)

(b) Hence use integration to find

$$\int_0^3 \sqrt{\left(\frac{x}{4-x}\right)}\,dx$$

giving your answer in the form $a\pi + b$, where a and b are exact constants.

(4)

4. **(Question 6 - C4 June 2015)**

6.

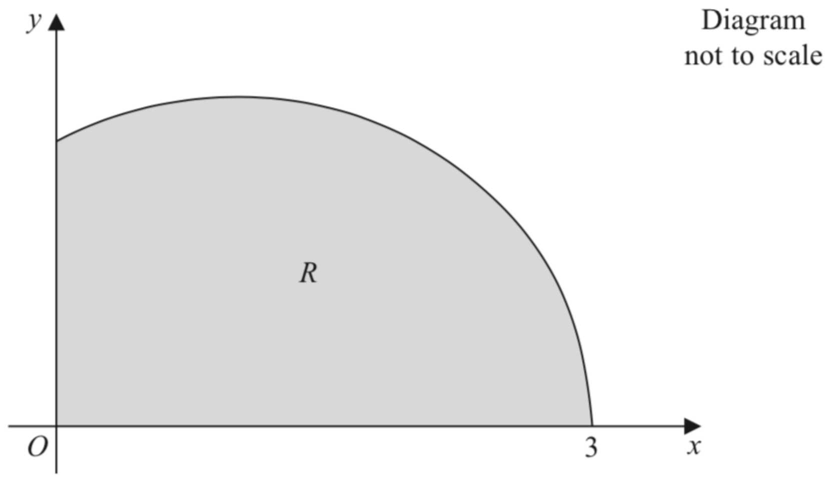

Figure 2

Figure 2 shows a sketch of the curve with equation $y = \sqrt{(3-x)(x+1)}$, $0 \leqslant x \leqslant 3$

The finite region R, shown shaded in Figure 2, is bounded by the curve, the x-axis, and the y-axis.

(a) Use the substitution $x = 1 + 2\sin\theta$ to show that

$$\int_0^3 \sqrt{(3-x)(x+1)}\,\mathrm{d}x = k\int_{-\frac{\pi}{6}}^{\frac{\pi}{2}} \cos^2\theta\,\mathrm{d}\theta$$

where k is a constant to be determined.

(5)

(b) Hence find, by integration, the exact area of R.

(3)

5. **(Question 3 - C4 June 2014)**

3.

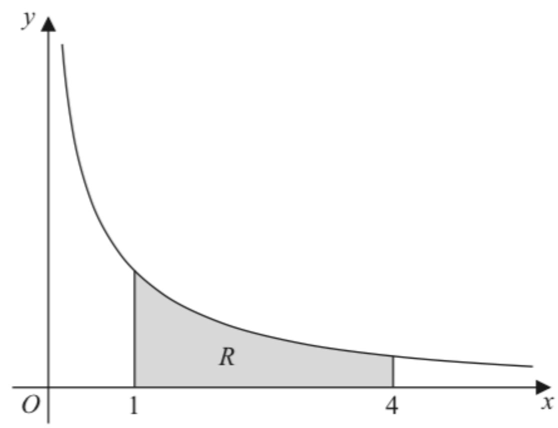

Figure 1

Figure 1 shows a sketch of part of the curve with equation $y = \dfrac{10}{2x + 5\sqrt{x}}$, $x > 0$

The finite region R, shown shaded in Figure 1, is bounded by the curve, the x-axis, and the lines with equations $x = 1$ and $x = 4$

The table below shows corresponding values of x and y for $y = \dfrac{10}{2x + 5\sqrt{x}}$

x	1	2	3	4
y	1.42857	0.90326		0.55556

(a) Complete the table above by giving the missing value of y to 5 decimal places.
(1)

(b) Use the trapezium rule, with all the values of y in the completed table, to find an estimate for the area of R, giving your answer to 4 decimal places.
(3)

(c) By reference to the curve in Figure 1, state, giving a reason, whether your estimate in part (b) is an overestimate or an underestimate for the area of R.
(1)

(d) Use the substitution $u = \sqrt{x}$, or otherwise, to find the exact value of

$$\int_1^4 \frac{10}{2x + 5\sqrt{x}}\,dx$$

(6)

6. **(Question 6 - C4 June 2014)**

6. (i) Find

$$\int x\,e^{4x}\,dx$$

(3)

(ii) Find

$$\int \frac{8}{(2x-1)^3}\,dx, \quad x > \frac{1}{2}$$

(2)

(iii) Given that $y = \frac{\pi}{6}$ at $x = 0$, solve the differential equation

$$\frac{dy}{dx} = e^x \operatorname{cosec} 2y \operatorname{cosec} y$$

(7)

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7. **(Question 5 - C4 June 2013)**

5. (a) Use the substitution $x = u^2$, $u > 0$, to show that

$$\int \frac{1}{x(2\sqrt{x}-1)}\,dx = \int \frac{2}{u(2u-1)}\,du$$

(3)

(b) Hence show that

$$\int_1^9 \frac{1}{x(2\sqrt{x}-1)}\,dx = 2\ln\left(\frac{a}{b}\right)$$

where a and b are integers to be determined.

(7)

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8. **(Question 4 - C4 June 2011)**

4.

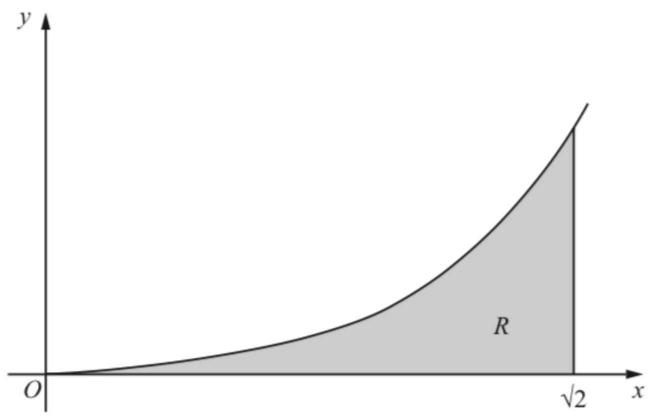

Figure 2

Figure 2 shows a sketch of the curve with equation $y = x^3 \ln(x^2 + 2)$, $x \geqslant 0$.
The finite region R, shown shaded in Figure 2, is bounded by the curve, the x-axis and the line $x = \sqrt{2}$.

The table below shows corresponding values of x and y for $y = x^3 \ln(x^2 + 2)$.

x	0	$\frac{\sqrt{2}}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{3\sqrt{2}}{4}$	$\sqrt{2}$
y	0		0.3240		3.9210

(a) Complete the table above giving the missing values of y to 4 decimal places.
(2)

(b) Use the trapezium rule, with all the values of y in the completed table, to obtain an estimate for the area of R, giving your answer to 2 decimal places.
(3)

(c) Use the substitution $u = x^2 + 2$ to show that the area of R is

$$\frac{1}{2}\int_2^4 (u - 2)\ln u \, du$$
(4)

(d) Hence, or otherwise, find the exact area of R.
(6)

9. **(Question 8 - C4 June 2011)**

8. (a) Find $\int (4y+3)^{-\frac{1}{2}} \, dy$

(2)

(b) Given that $y = 1.5$ at $x = -2$, solve the differential equation

$$\frac{dy}{dx} = \frac{\sqrt{(4y+3)}}{x^2}$$

giving your answer in the form $y = f(x)$.

(6)

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10. **(Question 2 - C4 June 2010)**

2. Using the substitution $u = \cos x + 1$, or otherwise, show that

$$\int_0^{\frac{\pi}{2}} e^{\cos x + 1} \sin x \, dx = e(e-1)$$

(6)

Solutions

1. (a) 1.86254
   (b) 1.6413
   (c) $a = 1$, $b = e$
   (d) $3\left(1 - \ln(e+2) + \ln(3)\right)$
2. (a) $\frac{1}{5}(2x-1)^{\frac{5}{2}} + c$
   (b) $k = \frac{17}{2}$
   (c) NOT EXAMINABLE
3. (a) $3\ln(3y+2) - 2\ln(y) + c$
   (b) i. $\lambda = 8$
      ii. $\frac{4}{3}\pi - \sqrt{3}$ so $a = \frac{4}{3}$ and $b = -\sqrt{3}$
4. (a) $k = 4$
   (b) $\frac{4}{3}\pi + \frac{\sqrt{3}}{2}$
5. (a) 0.68212
   (b) 2.5774
   (c) overestimate, the curve is convex
   (d) $10\ln\left(\frac{9}{7}\right)$
6. (a) $\frac{1}{4}xe^{4x} - \frac{1}{16}e^{4x} + c$
   (b) $\frac{-2}{(2x-1)^2} + c$
   (c) $\frac{1}{2}\sin(y) - \frac{1}{6}\sin(3y) = e^x - \frac{11}{12}$
7. (a) -
   (b) $2\ln\left(\frac{5}{3}\right)$ so $a = 5$, $b = 3$
8. (a) 0.0333, 1.3596
   (b) 1.30
   (c) -

(d) $\ln(2) + \frac{1}{2}$

9. (a) $\frac{1}{2}\sqrt{4y+3} + c$

(b) $y = \frac{1}{4}\left(2 - \frac{2}{x}\right)^2 - \frac{3}{4}$ or $y = \frac{x^2-8x+4}{4x^2}$

10. -

